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On the Zeros of the Riemann Zeta Function II

by

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§1. Introduction

We shall continue our study [5] concerning the distribution of the zeros of the Riemann zeta function $\zeta(s)$. We have been concerned with the high moments of the remainder term in the Riemann-von Mangoldt formula for the number of the zeros of $\zeta(s)$. The purpose of the present article is to give a mean value theorem for these high moments. Especially, we shall extend Selberg's results and refine Littlewood's results concerning this problem.

We start with recalling some of our previous results in Fujii [5]. We denote the non-trivial zeros of $\zeta(s)$ by $\rho = \beta + i\gamma$ with real numbers β and γ. Let T be a positive number. Let $N(T)$ denote the number of the zeros $\beta + i\gamma$ of $\zeta(s)$ in $0 < \gamma < T$, $0 < \beta < 1$, when $T \neq \gamma$ for any γ. When $T = \gamma$ for some γ, then we put

$$N(T) = \frac{1}{2}(N(T+0) + N(T-0)) .$$

Let

$$S(T) = \frac{1}{\pi}\arg\zeta\left(\frac{1}{2} + iT\right) \quad \text{for} \quad T \neq \gamma ,$$

where the argument is obtained by the continuous variation along the straight lines joining 2, $2 + iT$, and $\frac{1}{2} + iT$, starting with the value zero. When $T = \gamma$, then we put

$$S(T) = \frac{1}{2}(S(T+0) + S(T-0)) .$$

Then the well known Riemann-von Mangoldt formula (cf. p. 212 of Titchmarsh [14]) states that

$$N(T) = \frac{1}{\pi}\vartheta(T) + 1 + S(T) ,$$

where $\vartheta(T)$ is the continuous function defined by

$$\vartheta(T) = \Im\left(\log\Gamma\left(\frac{1}{4} + \frac{iT}{2}\right)\right) - \frac{1}{2}T\log\pi$$

with $\vartheta(0) = 0$, $\Gamma(s)$ being the gamma-function. It is well known that

$$\vartheta(T) = \frac{T}{2}\log\frac{T}{2\pi} - \frac{T}{2} - \frac{\pi}{8} + \frac{1}{48T} + \frac{7}{5760T^3} + \cdots$$

and that, for $T > T_o$, we have

$$S(T) = O(\log T)\,.$$

The last estimate was refined under the Riemann Hypothesis (R.H.) by Littlewood [10] and later by Selberg [13] in different ways as follows:

$$S(T) = O\left(\frac{\log T}{\log\log T}\right)\,.$$

Here we are concerned with the high moments of $S(T)$.

The first average is well known and classical. Littlewood [10] and Selberg [13] have shown that

$$\int_0^T S(t)dt = O(\log T)\,.$$

It seems to be difficult to go beyond this bound without assuming any unproved hypothesis. In fact, it is noticed on p. 335 of Titchmarsh [14] that

the Lindelöf hypothesis is equivalent to the statement that

$$\int_0^T S(t)dt = o(\log T) \quad (T \to \infty)\,.$$

If we assume the Riemann Hypothesis, then we have, due to Littlewood [10] and Selberg [13],

$$\int_0^T S(t)dt = O\biggl(\frac{\log T}{(\log\log T)^2}\biggr)\,.$$

At this stage there are two directions to consider the high moments of $S(t)$. One direction is to consider the high moments

$$\int_0^T S^{2k}(t)dt$$

for any integer $k \geq 1$. Another direction is to consider the multiple integrals of the form

$$\underbrace{\int_0^T \int_0^t \cdots \int_0^t}_{m} S(t)(dt)^m$$

for any integer $m \geq 1$.

In the first direction, Selberg [13] showed that for each integer $k \geq 1$,

$$\int_0^T S^{2k}(t)dt = \frac{(2k)!}{k!(2\pi)^{2k}} T(\log\log T)^k + O(T(\log\log T)^{k-\frac{1}{2}})\,.$$

This shows that the value distribution of $S(t)$ obeys the Gaussian law.

We recall that one of the important aspects of the study of $S(t)$ is a local behavior of the distribution of the zeros of $\zeta(s)$. In fact, we [1][2][3][4][6] have been concerned with the distribution of the zeros in short intervals and shown , in a latest form, that *for* $T > T_o$ *and for* $0 < \frac{2\pi\alpha}{\log\frac{T}{2\pi}} \ll T$

$$\int_0^T \left(S\left(t + \frac{2\pi\alpha}{\log\frac{T}{2\pi}}\right) - S(t)\right)^{2k} dt$$

$$= \begin{cases} \dfrac{(2k)!}{(2\pi)^{2k}k!} 2^k T(\log(2\pi\alpha) - Ci(2\pi\alpha) + C_o)^k \\ \qquad + O(T(Ak)^k((\log(2\pi\alpha) - Ci(2\pi\alpha) + C_o)^{k-\frac{1}{2}} + k^k)) \\ \qquad\qquad \textit{if} \quad 0 < \alpha \ll \log T \\ \dfrac{(2k)!}{(2\pi)^{2k}k!} 2^k T\left(\log\log T - \log\left|\zeta\left(1 + i\dfrac{2\pi\alpha}{\log\frac{T}{2\pi}}\right)\right|\right)^k \\ \qquad + O\left(T(Ak)^k\left(\left(\log\log T - \log\left|\zeta\left(1 + i\dfrac{2\pi\alpha}{\log\frac{T}{2\pi}}\right)\right|\right)^{k-\frac{1}{2}} + k^k\right)\right) \\ \qquad\qquad \textit{if} \quad \log T \ll \alpha \ll T\log T\ , \end{cases}$$

uniformly for an integer $k \geq 1$, where we put

$$Ci(x) = -\int_x^\infty \frac{\cos t}{t} dt,$$

C_o is the Euler constant and A is some positive absolute constant (cf. Theorem 2 on p. 23 of Fujii [6]).

When $0 < \alpha \ll 1$, this formula does not give an asymptotic formula. However, we can recover it, applying Goldston [9], for the case of $k = 1$, under the Riemann Hypothesis, as follows.

(*Under the Riemann Hypothesis*)
Suppose that $T > T_o$ and $0 < \frac{2\pi\alpha}{\log\frac{T}{2\pi}} \ll T$. Then we have

$$\int_0^T \left(S\left(t + \frac{2\pi\alpha}{\log\frac{T}{2\pi}}\right) - S(t)\right)^2 dt$$

$$= \begin{cases} \dfrac{T}{\pi^2}\left\{\displaystyle\int_0^{2\pi\alpha} \frac{1-\cos a}{a} da + \int_1^\infty \frac{F(a)}{a^2}(1 - \cos(2\pi\alpha a))\, da + o(1)\right\} \\ \qquad\qquad \textit{if} \quad 0 < \alpha = o(\log T) \\ \dfrac{T}{\pi^2}\left\{\log\log T - \log\left|\zeta\left(1 + i\dfrac{2\pi\alpha}{\log\frac{T}{2\pi}}\right)\right| + O(1)\right\} \\ \qquad\qquad \textit{if} \quad \log T \ll \alpha \ll T\log T\ , \end{cases}$$

where $F(a)$ is Montgomery's sum [12] *defined by*

$$F(a) \equiv F(a, T) \equiv \frac{1}{\frac{T}{2\pi}\log T} \sum_{0<\gamma,\gamma'\leq T} \left(\frac{T}{2\pi}\right)^{ia(\gamma-\gamma')} \frac{4}{4 + (\gamma - \gamma')^2}\ ,$$

γ and γ' running over the imaginary parts of the zeros of $\zeta(s)$, respectively.

It is shown by Goldston [9] that $\int_1^\infty \frac{F(a)}{a^2}(1-\cos(2\pi\alpha a))\,da$ is bounded. Furthermore, if we assume Montgomery's Conjecture [12] on $F(a)$, then we get a finer asymptotic formula for the case for $0<\alpha=o(\log T)$, which suggests a connection with the random matrix theory (cf. p. 247 of Fujii [4]).

We turn our attention to the second direction. To proceed further, we shall define first the high moments, explicitly, as follows. When $T\neq\gamma$, then we put

$$\tilde{S}_0(T)=S(T)$$

and

$$\tilde{S}_m(T)=\int_0^T \tilde{S}_{m-1}(t)dt+C_m$$

for any integer $m\geq 1$, where C_m's are the constants defined by

$$C_{2k-1}=(-1)^{k-1}\frac{1}{\pi}\underbrace{\int_{\frac{1}{2}}^\infty\int_\sigma^\infty\cdots\int_\sigma^\infty}_{2k-1}\log|\,\zeta(\sigma)\,|\,(d\sigma)^{2k-1}$$

for $m=2k-1$ and

$$C_{2k}=(-1)^{k-1}\underbrace{\int_{\frac{1}{2}}^1\int_\sigma^1\cdots\int_\sigma^1}_{2k}(d\sigma)^{2k}=(-1)^{k-1}\frac{1}{(2k)!\cdot 2^{2k}}$$

for $m=2k$, respectively. When $T=\gamma$, then we put

$$\tilde{S}_m(T)=\frac{1}{2}(\tilde{S}_m(T+0)+\tilde{S}_m(T-0)).$$

Concerning $\tilde{S}_m(T)$ for $m\geq 2$, Littlewood [10] and Selberg [13] have shown under the Riemann Hypothesis that

$$\tilde{S}_m(T)=O\left(\frac{\log T}{(\log\log T)^{m+1}}\right).$$

In Fujii [5], we have given a study of $\tilde{S}_m(T)$ without assuming any unproved hypothesis. To describe our results, we introduce some more notations. We define first the integral $I_m(T)$ as follows. When $T\neq\gamma$, then we put for $k\geq 1$

$$I_{2k-1}(T)=\frac{1}{\pi}(-1)^{k-1}\Re\left\{\underbrace{\int_{\frac{1}{2}}^\infty\int_\sigma^\infty\cdots\int_\sigma^\infty}_{2k-1}\log\zeta(\sigma+iT)\,(d\sigma)^{2k-1}\right\}$$

and

$$I_{2k}(T)=\frac{1}{\pi}(-1)^k\Im\left\{\underbrace{\int_{\frac{1}{2}}^\infty\int_\sigma^\infty\cdots\int_\sigma^\infty}_{2k}\log\zeta(\sigma+iT)\,(d\sigma)^{2k}\right\}.$$

When $T=\gamma$, then we put for $m\geq 1$

$$I_m(T) = \frac{1}{2}(I_m(T+0) + I_m(T-0)) .$$

For $\sigma \geq \frac{1}{2}$ and $T > T_o$, let $N(\sigma, T)$ be the number of the zeros $\beta + i\gamma$ of $\zeta(s)$ such that $\beta > \sigma$ and $0 < \gamma < T$ when $T \neq \gamma$. When $T = \gamma$, then we put

$$N(\sigma, T) = \frac{1}{2}((N(\sigma, T+0) + N(\sigma, T-0)) .$$

Then we have shown that there is a relation between $\tilde{S}_m(T)$ and $I_m(T)$ in the following form.

LEMMA 1. *Suppose that $T > 0$. Then we have*

$$\tilde{S}_1(T) = I_1(T)$$

and for any integer $m \geq 2$

$$\tilde{S}_m(T) = I_m(T) + 2 \sum_{\substack{h+2r=m \\ r\geq 1, h\geq 0}} (-1)^{r-1} \tilde{N}_{h,2r}(T) ,$$

where we put for $h \geq 1$ and $r \geq 1$

$$\tilde{N}_{h,2r}(T) = \underbrace{\int_0^T \cdots \int_0^t}_{h} \underbrace{\int_{\frac{1}{2}}^1 \int_\sigma^1 \cdots \int_\sigma^1}_{2r} N(\sigma, t)\,(d\sigma)^{2r}\,(d\,t)^h$$

and for $h = 0$ and $r \geq 1$

$$\tilde{N}_{0,2r}(T) = \underbrace{\int_{\frac{1}{2}}^1 \int_\sigma^1 \cdots \int_\sigma^1}_{2r} N(\sigma, T)\,(d\sigma)^{2r} .$$

The multiple integral $I_m(T)$ can be expressed as a single integral of the following form (cf. Lemma 2 on p. 4 in Fujii [5]): for any integer $m \geq 1$

$$I_m(T) = -\frac{1}{\pi}\Im\left\{\frac{i^m}{m!}\int_{\frac{1}{2}}^\infty \left(\sigma - \frac{1}{2}\right)^m \frac{\zeta'}{\zeta}(\sigma + iT)d\sigma\right\} .$$

From this expression, we have shown that

$$I_m(T) = O(\log T) \quad \text{for any integer } m \geq 1 ,$$

where the constant involved in the upper bound may depend on m (cf. Theorem 1 on p. 4 in Fujii [5]). This result for the case $m = 1$ was obtained by Littlewood [10] and Selberg [13].

Concerning the sum

$$2 \sum_{\substack{h+2r=m \\ r\geq 1, h\geq 0}} (-1)^{r-1} \tilde{N}_{h,2r}(T) ,$$

we can apply Selberg's density theorem (cf. p. 232 of Selberg [13]): for $T > T_o$ and some positive constant C,

$$N(\sigma, T) = O(T \log T \cdot e^{-C(\sigma-\frac{1}{2})\log T})$$

uniformly for $\sigma \geq \frac{1}{2}$. Thus we have

$$2 \sum_{\substack{h+2r=m \\ r\geq 1, h\geq 0}} (-1)^{r-1} \tilde{N}_{h,2r}(T) = O\left(\frac{T^{m-1}}{\log T}\right).$$

Consequently, we (cf. Theorem 2 on p.5 of Fujii [5]) have obtained that

$$\tilde{S}_m(T) = O\left(\frac{T^{m-1}}{\log T}\right) \quad \text{for any integer } m \geq 2\,.$$

A big gap between our result and the conditional result mentioned above leads, in fact, to the following fact (cf. Theorem 4 on p. 6 of Fujii [5]) .

The Riemann Hypothesis is equivalent to the statement that for any integer $m \geq 3$, *we have*

$$\tilde{S}_m(T) = o(T^{m-2}) \quad (T \to \infty)\,.$$

Here we shall rewrite Lemma 1 in the following form.

LEMMA 2. *Suppose that* $T > 0$. *For any integer* $m \geq 1$, *we have*

$$\tilde{S}_m(T) = I_m(T) + W_m(T)\,,$$

where we put

$$W_1(T) = 0$$

and for any $m \geq 2$,

$$W_m(T) = 2 \sum_{\substack{h+2r=m \\ r\geq 1, h\geq 0}} (-1)^{r-1} \frac{1}{(2r)! \cdot h!} \sum_{\substack{\beta+i\gamma \\ \beta>\frac{1}{2}, 0<\gamma<T}}{}' \left(\beta - \frac{1}{2}\right)^{2r} (T-\gamma)^h\,,$$

the dash denoting the halving convention as above when $h = 0$.

Now we are concerned with the following problem in the present article.

PROBLEM. *To study the high moments*

$$\int_0^T \tilde{S}_m^{2k}(t)dt \quad \textit{for each integer } m \geq 1 \textit{ and } k \geq 1\,.$$

Under the Riemann Hypothesis, Littlewood [11] (cf. Theorem 9 on p. 197 of [11]) showed that for each integer $m \geq 1$

$$\int_0^T \tilde{S}_m^2(t)dt \ll T\,.$$

Without assuming any unproved hypothesis, Selberg (cf. p. 255 of Selberg [13]) obtained an asymptotic formula

$$\int_0^T S_1^{2k}(t)dt = \frac{T}{\pi^{2k}} \sum_{\nu=0}^{2k} \binom{2k}{\nu} \Bigg\{ \frac{1}{2^{2k-\nu}} \sum_{\mu=0}^{2k-\nu} \binom{2k-\nu}{\mu}$$

$$\sum_{\substack{n_l=2 \\ n_1\cdots n_\mu = n_{\mu+1}\cdots n_{2k-\nu}}}^{\infty} \frac{\Lambda(n_1)\cdots\Lambda(n_{2k-\nu})}{n_1\cdots n_\mu (\log n_1 \cdots \log n_{2k-\nu})^2} \Bigg\}$$

$$\cdot \left\{ \int_{\frac{1}{2}}^{\infty} \log \mid \zeta(\sigma) \mid d\sigma \right\}^{\nu} + O\left(\frac{T}{\log T}\right),$$

where $\Lambda(n)$ is the von- Mangoldt function defined by

$$\Lambda(n) = \begin{cases} \log p & \text{if} \quad n = p^k \quad \text{with a prime } p \text{ and an integer } k \ge 1 \\ 0 & \text{otherwise} \end{cases}$$

and we put (cf. p 216 of Selberg [13])

$$S_1(T) = \int_0^T S(t)dt\,.$$

The purpose of the present article is to extend Selberg's result to $\tilde{S}_m(t)$ for any $m \ge 2$ and to get an explicit formula for the mean values

$$\int_0^T \tilde{S}_m^{2k}(t)dt$$

for each integer $m \ge 2$ and $k \ge 1$ without assuming any unproved hypothesis. We shall describe our results first for $k = 1$ and for any integer $m \ge 2$.

THEOREM 1. *Suppose that $T > T_o$. Then we have for any integer $m \ge 2$,*

$$\int_0^T \tilde{S}_m^2(t)dt = \frac{T}{2\pi^2} \sum_{n=2}^{\infty} \frac{\Lambda^2(n)}{n(\log n)^{2(m+1)}}$$

$$+4 \sum_{\substack{h+2r=m \\ r\ge 1, h\ge 0}} (-1)^{r-1} \frac{1}{(2r)!\cdot h!} \sideset{}{'}\sum_{\substack{\beta+i\gamma \\ \beta>\frac{1}{2}, 0<\gamma<T}} \left(\beta - \frac{1}{2}\right)^{2r}$$

$$\cdot \Bigg\{ \sum_{j=0}^{h-1} (-1)^j \frac{h!}{(h-j)!} I_{m+1+j}(T)(T-\gamma)^{h-j}$$

$$+ (-1)^h h! \cdot (I_{m+1+h}(T) - I_{m+1+h}(\gamma))$$

$$+ \sum_{j=0}^{h} (-1)^j \frac{h!}{(h-j)!} \cdot 2 \cdot \chi_0(m+j) \cdot (-1)^{\frac{m+j-1}{2}} H(m+1+j, h-j; T, \gamma) \Bigg\}$$

$$+4 \sum_{\substack{h_1+2r_1=m \\ r_1\geq 1, h_1\geq 0}} \sum_{\substack{h_2+2r_2=m \\ r_2\geq 1, h_2\geq 0}} (-1)^{r_1+r_2} \frac{1}{(2r_1)!\cdot h_1!} \frac{1}{(2r_2)!\cdot h_2!}$$

$$\cdot \sum_{\substack{\beta_1+i\gamma_1 \\ \beta_1>\frac{1}{2}, 0<\gamma_1<T}}{}^{'} \sum_{\substack{\beta_2+i\gamma_2 \\ \beta_2>\frac{1}{2}, 0<\gamma_2<T}}{}^{'} (\beta_1-\frac{1}{2})^{2r_1}\left(\beta_2-\frac{1}{2}\right)^{2r_2}$$

$$\cdot \sum_{l=0}^{h_1+h_2} (-1)^l \frac{T^{l+1}-(\max(\gamma_1,\gamma_2))^{l+1}}{l+1} \sum_{\substack{0\leq j\leq h_1, 0\leq k\leq h_2 \\ j+k=h_1+h_2-l}} \binom{h_1}{j}\binom{h_2}{k} \gamma_1^j \gamma_2^k$$

$$+ O\left(\frac{T}{\log T}\right),$$

where $\beta_1 + i\gamma_1$ and $\beta_2 + i\gamma_2$ run over the zeros of $\zeta(s)$, respectively, and we put

$$\chi_0(j) = \begin{cases} 0 & \textit{if } j \textit{ is even} \\ 1 & \textit{otherwise} \end{cases}$$

and for any $0 < V \leq T$,

$$H(m,0;T,V) = \tilde{N}_m(T) - \tilde{N}_m(V)$$

and for $l \geq 1$

$$H(m,l;T,V) = \tilde{N}_m(T)(T-V)^l - l\int_V^T (t-V)^{l-1}\tilde{N}_m(t)dt$$

with

$$\tilde{N}_m(T) = \underbrace{\int_{\frac{1}{2}}^1 \int_\sigma^1 \cdots \int_\sigma^1}_{m} N(\sigma,T)\,(d\sigma)^m .$$

This refines Littlewoods's result mentioned above for $m \geq 2$. For $m = 2$ the statement of Theorem 1 can be written down in a simpler form as in the following corollary.

COROLLARY. *Suppose that $T > T_o$. Then we have*

$$\int_0^T \tilde{S}_2^2(t)dt = \frac{T}{2\pi^2}\sum_{n=2}^\infty \frac{\Lambda^2(n)}{n(\log n)^6} + 2 \sum_{\substack{\beta+i\gamma \\ \beta>\frac{1}{2}, 0<\gamma<T}}{}^{'} \left(\beta-\frac{1}{2}\right)^2 (I_3(T)-I_3(\gamma))$$

$$+ \sum_{\substack{\beta+i\gamma, \beta'+i\gamma' \\ \beta>\frac{1}{2}, \beta'>\frac{1}{2}, 0<\gamma,\gamma'<T}}{}^{'} \left(\beta-\frac{1}{2}\right)^2\left(\beta'-\frac{1}{2}\right)^2 (T-\max(\gamma,\gamma'))$$

$$+ O\left(\frac{T}{\log T}\right),$$

where $\beta + i\gamma$ and $\beta' + i\gamma'$ run over the zeros of $\zeta(s)$, respectively, $I_3(T)$ is defined above and since $I_3(T) \ll \log T$, we have

$$2 \sum_{\substack{\beta+i\gamma \\ \beta>\frac{1}{2}, 0<\gamma<T}}{}' \left(\beta - \frac{1}{2}\right)^2 (I_3(T) - I_3(\gamma)) \ll T\,.$$

Under the Riemann Hypothesis, the second and the third terms disappear in both Theorem 1 and Corollary. In stead of describing explicitly the mean values of $\int_0^T \tilde{S}_m^{2k}(t)dt$, we shall state the following result as a theorem. In fact, we shall use it in the course of the proof of Theorem 1.

THEOREM 2. *Suppose that $T > T_o$. Then for any integer $m \geq 2$ and for any integer $k \geq 1$, we have*

$$\int_0^T (\tilde{S}_m(t) - W_m(t))^{2k} dt$$
$$= \frac{T}{(2\pi)^{2k}} \sum_{j=0}^{2k} \binom{2k}{j} (-1)^{(j+k)(m+1)}$$
$$\cdot \sum_{\substack{n_l=2 \\ n_1\cdots n_j = n_{j+1}\cdots n_{2k}}}^{\infty} \frac{\Lambda(n_1)\cdots\Lambda(n_{2k})}{n_1\cdots n_j(\log n_1 \cdots \log n_{2k})^{m+1}} + O\left(\frac{T}{\log T}\right).$$

Consequently, we get the following theorem under the Riemann Hypothesis.

THEOREM 3 (On the Riemann Hypothesis). *Suppose that $T > T_o$. Then for any integer $m \geq 2$ and for any integer $k \geq 1$, we have*

$$\int_0^T \tilde{S}_m^{2k}(t)dt$$
$$= \frac{T}{(2\pi)^{2k}} \sum_{j=0}^{2k} \binom{2k}{j} (-1)^{(j+k)(m+1)}$$
$$\cdot \sum_{\substack{n_l=2 \\ n_1\cdots n_j = n_{j+1}\cdots n_{2k}}}^{\infty} \frac{\Lambda(n_1)\cdots\Lambda(n_{2k})}{n_1\cdots n_j(\log n_1 \cdots \log n_{2k})^{m+1}} + O\left(\frac{T}{\log T}\right).$$

This refines again Littlewood's result mentioned above.

In the following sections, we shall give the details of the proof of Theorems 1 and 2.

§2. Proof of Theorem 1

We apply the explicit formula for $\tilde{S}_m(T)$ as described in Lemma 2 of the introduction. We have, at the first step,

$$\int_0^T \tilde{S}_m^2(t)dt = \int_0^T I_m^2(t)dt + 2\int_0^T I_m(t)W_m(t)dt + \int_0^T W_m^2(t)dt$$
$$= U_1 + U_2 + U_3\,, \quad \text{say}\,.$$

We notice first that U_3 is written explicitly as follows.

$$U_3 = \int_0^T \Bigg\{ 2 \sum_{\substack{h_1+2r_1=m \\ r_1\ge 1, h_1\ge 0}} (-1)^{r_1-1} \frac{1}{(2r_1)!\cdot h_1!} \sum\nolimits'_{\substack{\beta_1+i\gamma_1 \\ \beta_1>\frac{1}{2}, 0<\gamma_1<t}} \left(\beta_1 - \frac{1}{2}\right)^{2r_1} (t-\gamma_1)^{h_1} \Bigg\}$$
$$\cdot \Bigg\{ 2 \sum_{\substack{h_2+2r_2=m \\ r_2\ge 1, h_2\ge 0}} (-1)^{r_2-1} \frac{1}{(2r_2)!\cdot h_2!} \sum\nolimits'_{\substack{\beta_2+i\gamma_2 \\ \beta_2>\frac{1}{2}, 0<\gamma_2<t}} \left(\beta_2 - \frac{1}{2}\right)^{2r_2} (t-\gamma_2)^{h_2} \Bigg\} dt$$
$$= 4 \sum_{\substack{h_1+2r_1=m \\ r_1\ge 1, h_1\ge 0}} \sum_{\substack{h_2+2r_2=m \\ r_2\ge 1, h_2\ge 0}} (-1)^{r_1-1} \frac{1}{(2r_1)!\cdot h_1!} (-1)^{r_2-1} \frac{1}{(2r_2)!\cdot h_2!}$$
$$\cdot \sum\nolimits'_{\substack{\beta_1+i\gamma_1 \\ \beta_1>\frac{1}{2}, 0<\gamma_1<T}} \sum\nolimits'_{\substack{\beta_2+i\gamma_2 \\ \beta_2>\frac{1}{2}, 0<\gamma_2<T}} \left(\beta_1 - \frac{1}{2}\right)^{2r_1} \left(\beta_2 - \frac{1}{2}\right)^{2r_2}$$
$$\cdot \int_{\max(\gamma_1,\gamma_2)}^T (t-\gamma_1)^{h_1} (t-\gamma_2)^{h_2} dt$$
$$= 4 \sum_{\substack{h_1+2r_1=m \\ r_1\ge 1, h_1\ge 0}} \sum_{\substack{h_2+2r_2=m \\ r_2\ge 1, h_2\ge 0}} (-1)^{r_1+r_2} \frac{1}{(2r_1)!\cdot h_1!} \frac{1}{(2r_2)!\cdot h_2!}$$
$$\cdot \sum\nolimits'_{\substack{\beta_1+i\gamma_1 \\ \beta_1>\frac{1}{2}, 0<\gamma_1<T}} \sum\nolimits'_{\substack{\beta_2+i\gamma_2 \\ \beta_2>\frac{1}{2}, 0<\gamma_2<T}} \left(\beta_1 - \frac{1}{2}\right)^{2r_1} \left(\beta_2 - \frac{1}{2}\right)^{2r_2}$$
$$\cdot \sum_{l=0}^{h_1+h_2} (-1)^l \frac{T^{l+1} - (\max(\gamma_1,\gamma_2))^{l+1}}{l+1} \sum_{\substack{0\le j\le h_1, 0\le k\le h_2 \\ j+k=h_1+h_2-l}} \binom{h_1}{j}\binom{h_2}{k} \gamma_1^j \gamma_2^k\,.$$

Next, since

$$U_2 = 4\cdot \sum_{\substack{h+2r=m \\ r\ge 1, h\ge 0}} (-1)^{r-1} \frac{1}{(2r)!\cdot h!} \sum\nolimits'_{\substack{\beta+i\gamma \\ \beta>\frac{1}{2}, 0<\gamma<T}} \left(\beta - \frac{1}{2}\right)^{2r} \int_\gamma^T I_m(t)(t-\gamma)^h dt\,,$$

we need to evaluate the integral

$$\int_{\gamma}^{T} I_m(t)(t-\gamma)^h dt\,.$$

We shall prove the following lemma.

LEMMA 3. *Suppose that $T > T_o$ and $0 < V \le T$. Then we have for any integer $m \ge 1$ and for any integer $h \ge 0$,*

$$\int_V^T I_m(t)(t-V)^h dt = \sum_{j=0,h\ge 1}^{h-1} (-1)^j \frac{h!}{(h-j)!} I_{m+1+j}(T)(T-V)^{h-j}$$

$$+ (-1)^h h! \cdot (I_{m+1+h}(T) - I_{m+1+h}(V))$$

$$+ \sum_{j=0}^{h} (-1)^j \frac{h!}{(h-j)!} 2 \cdot \chi_0(m+j) \cdot (-1)^{\frac{m+j-1}{2}} H(m+1+j, h-j; T, V)\,,$$

where $\chi_0(m+j)$ and $H(m+1+j, h-j; T, V)$ are defined in the statement of Theorem 1.

Proof. We shall prove this by induction on h. We shall treat first the case for $h = 0$, Namely, we shall evaluate the integral

$$\int_V^T I_m(t)dt\,.$$

When $m = 2k-1$, then by the definition of $I_{2k-1}(t)$, we get

$$\int_V^T I_m(t)dt = \frac{1}{\pi}(-1)^{k-1} \int_V^T \underbrace{\int_{\frac{1}{2}}^{\infty} \int_{\sigma}^{\infty} \cdots \int_{\sigma}^{\infty}}_{2k-1} \Re\{\log \zeta(\sigma+it)\}\,(d\sigma)^{2k-1} dt$$

$$= \frac{1}{\pi}(-1)^{k-1} \underbrace{\int_{\frac{1}{2}}^{\infty} \int_{\sigma}^{\infty} \cdots \int_{\sigma}^{\infty}}_{2k-1} \int_V^T \Re\{\log \zeta(\sigma+it)\}\,dt\,(d\sigma)^{2k-1}\,,$$

where the change of the integrals can be justified as in pp. 10–14 of Fujii [5]. Here we notice that for $T \ne \gamma$ and for $\sigma \ge \frac{1}{2}$, we have, by p. 300 of Littlewood [10] (and also pp. 220-221 of Titchmarsh [14]),

$$2\pi \int_{\sigma}^{1} \nu(\sigma, T)d\sigma = \Im\left\{\int_{\sigma}^{\infty} \log \zeta(\sigma+iT)\,d\sigma\right\} + \int_0^T \log \mid \zeta(\sigma+it) \mid dt,$$

where we put

$$\nu(\sigma, T) = \begin{cases} N(\sigma, T) - \frac{1}{2} & \text{if } \sigma < 1 \\ 0 & \text{if } \sigma \ge 1\,, \end{cases}$$

$N(\sigma, T)$ being defined in the introduction. When $T = \gamma$, then the relations hold by halving conventions as above. Thus we have

$$\begin{aligned}
&\int_V^T I_m(t)dt \\
&\quad = \frac{1}{\pi}(-1)^{k-1} \underbrace{\int_{\frac{1}{2}}^{\infty}\int_{\sigma}^{\infty}\cdots\int_{\sigma}^{\infty}}_{2k-1} \Big\{ 2\pi \int_{\sigma}^{1} (\nu(\sigma, T) - \nu(\sigma, V))d\sigma \\
&\quad\quad - \Im\Big\{ \int_{\sigma}^{\infty} (\log\zeta(\sigma + iT) - \log\zeta(\sigma + iV))\, d\sigma \Big\}\Big\}\, (d\sigma)^{2k-1} \\
&\quad = 2\cdot(-1)^{k-1} \underbrace{\int_{\frac{1}{2}}^{\infty}\int_{\sigma}^{\infty}\cdots\int_{\sigma}^{\infty}\int_{\sigma}^{1}}_{2k} (N(\sigma, T) - N(\sigma, V))\, (d\sigma)^{2k} \\
&\quad\quad + \frac{1}{\pi}(-1)^{k}\Im\Big\{ \underbrace{\int_{\frac{1}{2}}^{\infty}\int_{\sigma}^{\infty}\cdots\int_{\sigma}^{\infty}\int_{\sigma}^{\infty}}_{2k} (\log\zeta(\sigma + iT) \\
&\quad\quad - \log\zeta(\sigma + iV))\ (d\sigma)^{2k} \Big\} \\
&\quad = 2\cdot(-1)^{k-1}(\tilde{N}_{2k}(T) - \tilde{N}_{2k}(V)) + I_{2k}(T) - I_{2k}(V) \\
&\quad = 2\cdot(-1)^{\frac{m-1}{2}}(\tilde{N}_{m+1}(T) - \tilde{N}_{m+1}(V)) + I_{m+1}(T) - I_{m+1}(V) \\
&\quad = 2\cdot\chi_0(m)\cdot(-1)^{\frac{m-1}{2}} H(m+1, 0; T, V) + I_{m+1}(T) - I_{m+1}(V)\,,
\end{aligned}$$

where $\tilde{N}_{2k}(T)$ is defined in the statement of Theorem 1. We shall next treat the case for $m = 2k$. In this case we shall use the following relation for $T \neq \gamma$ and for $\sigma \geq \frac{1}{2}$,

$$\Im\Big\{ \int_0^T \log\zeta(\sigma + it)dt \Big\} = \int_{\sigma}^{\infty} \log \mid \zeta(\sigma + iT) \mid\, d\sigma - \int_{\sigma}^{\infty} \log \mid \zeta(\sigma) \mid\, d\sigma\,,$$

where for $T = \gamma$ the relation hold by halving conventions as above. Now by the definition of $I_{2k}(T)$, we get

$$\begin{aligned}
\int_V^T I_m(t)dt &= \frac{1}{\pi}(-1)^{k}\Im\Big\{ \int_V^T \underbrace{\int_{\frac{1}{2}}^{\infty}\int_{\sigma}^{\infty}\cdots\int_{\sigma}^{\infty}}_{2k} \log\zeta(\sigma + iT)\, (d\sigma)^{2k} dt \Big\} \\
&= \frac{1}{\pi}(-1)^{k} \underbrace{\int_{\frac{1}{2}}^{\infty}\int_{\sigma}^{\infty}\cdots\int_{\sigma}^{\infty}}_{2k} \int_V^T \Im\{\log\zeta(\sigma + it)\}dt\, (d\sigma)^{2k}
\end{aligned}$$

$$= \frac{1}{\pi}(-1)^k \underbrace{\int_{\frac{1}{2}}^{\infty}\int_{\sigma}^{\infty}\cdots\int_{\sigma}^{\infty}}_{2k}\left(\int_{\sigma}^{\infty}(\log|\zeta(\sigma+iT)|-\log|\zeta(\sigma+iV)|\right)d\sigma(d\sigma)^{2k}$$

$$= I_{2k+1}(T) - I_{2k+1}(V) = I_{m+1}(T) - I_{m+1}(V)\,.$$

These prove the assertion for $h = 0$.

Now suppose that the assertion is correct for h. We shall evaluate the integral for $h + 1$.

$$\begin{aligned}\int_V^T I_m(t)(t-V)^{h+1}dt \\ &= (h+1)\int_V^T I_m(t)\int_0^{t-V} u^h\,dudt \\ &= (h+1)\int_0^{T-V} u^h\,du\int_{u+V}^T I_m(t)dt.\end{aligned}$$

Applying the assertion for $h = 0$ to the last integral, we get

$$\begin{aligned}&\int_V^T I_m(t)(t-V)^{h+1}dt \\ &= (h+1)\int_0^{T-V} u^h\{2\cdot\chi_0(m)\cdot(-1)^{\frac{m-1}{2}}(\tilde{N}_{m+1}(T)-\tilde{N}_{m+1}(u+V)) \\ &\quad + I_{m+1}(T) - I_{m+1}(u+V)\}\,du \\ &= 2\cdot\chi_0(m)\cdot(-1)^{\frac{m-1}{2}}\left\{\tilde{N}_{m+1}(T)(T-V)^{h+1}-(h+1)\int_V^T(t-V)^h\tilde{N}_{m+1}(t)dt\right\} \\ &\quad + I_{m+1}(T)(T-V)^{h+1}-(h+1)\int_V^T(t-V)^h I_{m+1}(t)dt \\ &= 2\cdot\chi_0(m)\cdot(-1)^{\frac{m-1}{2}}H(m+1,h+1;T,V)+I_{m+1}(T)(T-V)^{h+1} \\ &\quad -(h+1)\int_V^T(t-V)^h I_{m+1}(t)dt\,.\end{aligned}$$

By the induction hypothesis, we get further

$$\begin{aligned}&\int_V^T I_m(t)(t-V)^{h+1}dt \\ &= 2\cdot\chi_0(m)\cdot(-1)^{\frac{m-1}{2}}H(m+1,h+1;T,V)+I_{m+1}(T)(T-V)^{h+1} \\ &\quad -(h+1)\Bigg\{\sum_{j=0,h\geq1}^{h-1}(-1)^j\frac{h!}{(h-j)!}I_{m+2+j}(T)(T-V)^{h-j} \\ &\quad +(-1)^h h!(I_{m+2+h}(T)-I_{m+2+h}(V))\end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_{j=0}^{h} (-1)^j \frac{h!}{(h-j)!} 2 \cdot \chi_0(m+1+j) \cdot (-1)^{\frac{m+j}{2}} H(m+2+j, h-j; T, V) \Big\} \\
&= I_{m+1}(T)(T-V)^{h+1} \\
&\quad + \sum_{j=0, h \geq 1}^{h-1} (-1)^{j+1} \frac{(h+1)!}{(h-j)!} I_{m+2+j}(T)(T-V)^{h-j} \\
&\quad + 2 \cdot \chi_0(m) \cdot (-1)^{\frac{m-1}{2}} H(m+1, h+1; T, V) \\
&\quad + (-1)^{h+1}(h+1)!(I_{m+2+h}(T) - I_{m+2+h}(V)) \\
&\quad + \sum_{j=0}^{h} (-1)^{j+1} \frac{(h+1)!}{(h-j)!} 2 \cdot \chi_0(m+1+j) \cdot (-1)^{\frac{m+j}{2}} H(m+2+j, h-j; T, V) \\
&= I_{m+1}(T)(T-V)^{h+1} \\
&\quad + \sum_{j=1, h \geq 1}^{h} (-1)^j \frac{(h+1)!}{(h+1-j)!} I_{m+1+j}(T)(T-V)^{h+1-j} \\
&\quad + 2 \cdot \chi_0(m) \cdot (-1)^{\frac{m-1}{2}} H(m+1, h+1; T, V) \\
&\quad + (-1)^{h+1}(h+1)!(I_{m+2+h}(T) - I_{m+2+h}(V)) \\
&\quad + \sum_{j=1}^{h+1} (-1)^j \frac{(h+1)!}{(h+1-j)!} 2\chi_0(m+j)(-1)^{\frac{m+j-1}{2}} H(m+1+j, h+1-j; T, V) .
\end{aligned}
$$

Thus we get

$$
\begin{aligned}
&\int_V^T I_m(t)(t-V)^{h+1} dt \\
&\quad = \sum_{j=0, h \geq 1}^{h} (-1)^j \frac{(h+1)!}{(h+1-j)!} I_{m+1+j}(T)(T-V)^{h+1-j} \\
&\quad + (-1)^{h+1}(h+1)!(I_{m+2+h}(T) - I_{m+2+h}(V)) \\
&\quad + \sum_{j=1}^{h+1} (-1)^j \frac{(h+1)!}{(h+1-j)!} 2\chi_0(m+j)(-1)^{\frac{m+j-1}{2}} H(m+1+j, h+1-j; T, V) .
\end{aligned}
$$

This is our assertion for $h+1$. Q.E.D.

Using this lemma, we can write down U_2 explicitly as follows.

$$
U_2 = 4 \sum_{\substack{h+2r=m \\ r \geq 1, h \geq 0}} (-1)^{r-1} \frac{1}{(2r)! \cdot h!} \sum_{\substack{\beta+i\gamma \\ \beta > \frac{1}{2}, 0 < \gamma < T}}{}' \left(\beta - \frac{1}{2}\right)^{2r} \int_\gamma^T I_m(t)(t-\gamma)^h dt
$$

$$
\begin{aligned}
&= 4 \sum_{\substack{h+2r=m \\ r\geq 1, h\geq 0}} (-1)^{r-1} \frac{1}{(2r)! \cdot h!} \sideset{}{'}\sum_{\substack{\beta+i\gamma \\ \beta>\frac{1}{2}, 0<\gamma<T}} \left(\beta - \frac{1}{2}\right)^{2r} \\
&\quad \cdot \left\{ \sum_{j=0, h\geq 1}^{h-1} (-1)^j \frac{h!}{(h-j)!} I_{m+1+j}(T)(T-\gamma)^{h-j} \right. \\
&\quad + (-1)^h h! (I_{m+1+h}(T) - I_{m+1+h}(\gamma)) \\
&\quad \left. + \sum_{j=0}^{h} (-1)^j \frac{(h)!}{(h-j)!} 2 \cdot \chi_0(m+j) \cdot (-1)^{\frac{m+j-1}{2}} H(m+1+j, h+1-j; T, \gamma) \right\}.
\end{aligned}
$$

The evaluation of U_1 is a consequence of the proof of Theorem 2, which will be given in the next section.

§3. Proof of Theorem 2 and completion of the proof of Theorem 1.

In this section we shall prove Theorem 2. We suppose that $m \geq 2$. For this purpose, we introduce some notations first (cf. p. 233 and p.235 of Selberg [13]). Let $s = \sigma + it$ with $\sigma \geq \frac{1}{2}$ and $t \geq 2$. Let X be a positive number satisfying $4 \leq X \leq t^2$. We put

$$
\sigma_{X,t} = \frac{1}{2} + 2 \max_{\rho} \left(\beta - \frac{1}{2}, \frac{2}{\log X} \right),
$$

where ρ runs through all zeros $\beta + i\gamma$ of $\zeta(s)$ for which

$$
| t - \gamma | \leq \frac{X^{3|\beta - \frac{1}{2}|}}{\log X}.
$$

We put

$$
\Lambda_X(n) = \begin{cases}
\Lambda(n) & \text{for} \quad 1 \leq n \leq X \\
\Lambda(n) \dfrac{\log^2 \frac{X^3}{n} - 2\log^2 \frac{X^2}{n}}{2\log^2 X} & \text{for} \quad X \leq n \leq X^2 \\
\Lambda(n) \dfrac{\log^2 \frac{X^3}{n}}{2\log^2 X} & \text{for} \quad X^2 \leq n \leq X^3 .
\end{cases}
$$

Now we shall use the following explicit formula for $I_m(T)$ (cf. p. 17 of Fujii [5]).

LEMMA 4. *Suppose that $t \geq 2$ and $4 \leq X \leq t^2$. Then we have for any integer $m \geq 1$,*

$$I_m(t) = \frac{1}{\pi}\Im\Bigg\{i^m \sum_{n<X^3} \frac{\Lambda_X(n)}{n^{\sigma_{X,t}+it}(\log n)^{m+1}} \sum_{\nu=0}^{m} \frac{((\sigma_{X,t}-\frac{1}{2})\log n)^\nu}{\nu!}\Bigg\}$$
$$+ O\bigg(\Big(\sigma_{X,t}-\frac{1}{2}\Big)^{m+1}\bigg|\sum_{n<X^3}\frac{\Lambda_X(n)}{n^{\sigma_{X,t}+it}}\bigg|\bigg) + O\bigg(\Big(\sigma_{X,t}-\frac{1}{2}\Big)^{m+1}\log t\bigg).$$

We denote an arbitrarily small positive constant by ε. We put $X = T^{\frac{a}{k}}$ with an appropriate positive constant a, which may depend on ε. We put also $T_1 = \sqrt{X}$. We decompose the first term of $I_m(T)$ in the above lemma as follows.

$$\frac{1}{\pi}\Im\Bigg\{i^m \sum_{n<X^3} \frac{\Lambda_X(n)}{n^{\sigma_{X,t}+it}(\log n)^{m+1}} \sum_{\nu=0}^{m} \frac{((\sigma_{X,t}-\frac{1}{2})\log n)^\nu}{\nu!}\Bigg\}$$
$$= \frac{1}{\pi}\Im\Bigg\{i^m \sum_{n<X^3} \frac{\Lambda(n)}{n^{\frac{1}{2}+it}(\log n)^{m+1}}\Bigg\}$$
$$+ \sum_{\nu=1}^{m} \frac{(\sigma_{X,t}-\frac{1}{2})^\nu}{\nu!}\frac{1}{\pi}\Im\Bigg\{i^m \sum_{n<X^3} \frac{\Lambda(n)}{n^{\frac{1}{2}+it}(\log n)^{m+1-\nu}}\Bigg\}$$
$$- \sum_{\nu=0}^{m} \frac{(\sigma_{X,t}-\frac{1}{2})^\nu}{\nu!}\frac{1}{\pi}\Im\Bigg\{i^m \sum_{n<X^3} \frac{\Lambda(n)-\Lambda_X(n)}{n^{\frac{1}{2}+it}(\log n)^{m+1-\nu}}\Bigg\}$$
$$+ \sum_{\nu=0}^{m} \frac{(\sigma_{X,t}-\frac{1}{2})^\nu}{\nu!}\frac{1}{\pi}\Im\Bigg\{i^m \sum_{n<X^3} \frac{\Lambda_X(n)}{n^{it}(\log n)^{m+1-\nu}}\bigg(\frac{1}{n^{\sigma_{X,t}}}-\frac{1}{\sqrt{n}}\bigg)\Bigg\}.$$

Hence, we get

$$\int_{T_1}^{T}\Bigg(I_m(t) - \frac{1}{\pi}\Im\Bigg\{i^m \sum_{n<X^3} \frac{\Lambda(n)}{n^{\frac{1}{2}+it}(\log n)^{m+1}}\Bigg\}\Bigg)^{2k} dt$$
$$\ll \sum_{\nu=1}^{m}\int_0^T \Big(\sigma_{X,t}-\frac{1}{2}\Big)^{2k\nu}\bigg|\sum_{n<X^3}\frac{\Lambda(n)}{n^{\frac{1}{2}+it}(\log n)^{m+1-\nu}}\bigg|^{2k} dt$$
$$+ \sum_{\nu=0}^{m}\int_0^T \Big(\sigma_{X,t}-\frac{1}{2}\Big)^{2k\nu}\bigg|\sum_{n<X^3}\frac{\Lambda(n)-\Lambda_X(n)}{n^{\frac{1}{2}+it}(\log n)^{m+1-\nu}}\bigg|^{2k} dt$$
$$+ \sum_{\nu=0}^{m}\int_0^T \Big(\sigma_{X,t}-\frac{1}{2}\Big)^{2k\nu}\bigg|\sum_{n<X^3}\frac{\Lambda_X(n)}{n^{it}(\log n)^{m+1-\nu}}\bigg(\frac{1}{n^{\sigma_{X,t}}}-\frac{1}{\sqrt{n}}\bigg)\bigg|^{2k} dt$$
$$+ \int_0^T \Big(\sigma_{X,t}-\frac{1}{2}\Big)^{2k(m+1)}\bigg|\sum_{n<X^3}\frac{\Lambda_X(n)}{n^{\sigma_{X,t}+it}}\bigg|^{2k} dt$$

$$+\int_0^T \left(\sigma_{X,t}-\frac{1}{2}\right)^{2k(m+1)} \log^{2k} t\, dt$$
$$=\sum_{\nu=1}^{m} U_{4,\nu}+\sum_{\nu=0}^{m} U_{5,\nu}+\sum_{\nu=0}^{m} U_{6,\nu}+U_7+U_8\,, \quad \text{say}\,.$$

We notice the following lemma which is Lemma 12 of Selberg [13].

LEMMA 5.
If $T > T_o$, $X \ge 2$, $1 \le \xi \le X^8$, $X^3\xi^2 \le T^{\frac{1}{8}}$, we have for any $0 \le \nu \le 8k$,

$$\int_0^T \left(\sigma_{X,t}-\frac{1}{2}\right)^{\nu} \xi^{\sigma_{X,t}-\frac{1}{2}} dt = O\left(\frac{T}{(\log X)^{\nu}}\right).$$

Thus we get first

$$U_8 \ll \frac{T}{(\log X)^{2km}}\,.$$

We get next

$$\sum_{\nu=1}^{m} U_{4,\nu} \ll \sum_{\nu=1}^{m}\left(\int_0^T \left(\sigma_{X,t}-\frac{1}{2}\right)^{4k\nu} dt\right)^{\frac{1}{2}}\left(\int_0^T \left|\sum_{n<X^3} \frac{\Lambda(n)}{n^{\frac{1}{2}+it}(\log n)^{m+1-\nu}}\right|^{4k} dt\right)^{\frac{1}{2}}$$
$$\ll \sum_{\nu=1}^{m} \frac{\sqrt{T}}{(\log X)^{2k\nu}}\left\{\left(\int_0^T \left|\sum_{p<X^3} \frac{\log p}{p^{\frac{1}{2}+it}(\log p)^{m+1-\nu}}\right|^{4k} dt\right)^{\frac{1}{2}}\right.$$
$$\left.+\left(\int_0^T \left|\sum_{p^2<X^3} \frac{\log p}{p^{1+i2t}(\log p)^{m+1-\nu}}\right|^{4k} dt\right)^{\frac{1}{2}}+\sqrt{T}\right\}.$$

We notice that

$$\int_0^T \left|\sum_{p<X^3} \frac{a_1(p)}{p^{\frac{1}{2}+it}}\right|^{4k} dt = \int_0^T \left|\sum_{n<X^{6k}} \frac{b_1(n)}{n^{\frac{1}{2}+it}}\right|^{2} dt$$
$$\ll \sum_{n<X^{6k}} \frac{|\, b_1(n)\,|^2}{n}(T+O(n))\,,$$

where we put

$$a_1(p)=\frac{1}{(\log p)^{m-\nu}}$$

and

$$b_1(n)=\sum_{p_1p_2\cdots p_{2k}=n} a_1(p_1)a_1(p_2)\cdots a_1(p_{2k})\,.$$

Since

$$b_1(n) \ll 1\,,$$

we get

$$\sum_{n<X^{6k}} \frac{|\, b_1(n)\, |^2}{n}(T+O(n)) \ll \sum_{n<X^{6k}} \frac{|\, b_1(n)\, |}{n}(T+O(n))$$

$$\ll T\Big(\sum_{p<X^3} \frac{a_1(p)}{p}\Big)^{2k} + \Big(\sum_{p<X^3} a_1(p)\Big)^{2k} \ll T\cdot \begin{cases} (\log\log X)^{2k} & \text{if } \nu=m \\ 1 & \text{otherwise.} \end{cases}$$

In the same manner, we get

$$\int_0^T \Big|\sum_{p^2<X^3} \frac{\log p}{p^{1+i2t}(\log p)^{m+1-\nu}}\Big|^{4k} dt \ll T\,.$$

Hence, we get

$$\sum_{\nu=1}^{m} U_{4,\nu} \ll \sum_{\nu=1}^{m-1} \frac{\sqrt{T}}{(\log X)^{2k\nu}}\sqrt{T} + \frac{\sqrt{T}}{(\log X)^{2km}}\sqrt{T}(\log\log X)^k$$

$$\ll \frac{T}{(\log X)^{2k}} + \frac{T(\log\log X)^k}{(\log X)^{2km}} \ll \frac{T}{(\log X)^{2k}}\,.$$

Next, we have

$$U_{5,0} \ll \int_0^T \Big|\sum_{p<X^3} \frac{\Lambda(p)-\Lambda_X(p)}{p^{\frac{1}{2}+it}(\log p)^{m+1}}\Big|^{2k} dt$$

$$+ \int_0^T \Big|\sum_{p^2<X^3} \frac{\Lambda(p^2)-\Lambda_X(p^2)}{p^{1+i2t}(\log p)^{m+1}}\Big|^{2k} dt$$

$$+ \int_0^T \Big|\sum_{p^r<X^3, r\ge 3} \frac{\Lambda(p^r)-\Lambda_X(p^r)}{p^{\frac{r}{2}+irt}(\log p)^{m+1}}\Big|^{2k} dt\,.$$

The first integral is

$$= \int_0^T \Big|\sum_{X^k<n<X^{3k}} \frac{b_2(n)}{n^{\frac{1}{2}+it}}\Big|^2 dt \ll \sum_{X^k<n<X^{3k}} \frac{|b_2(n)|^2}{n}(T+O(n))$$

$$\ll \frac{1}{(\log X)^{mk}}\Big\{T\Big(\sum_{p<X^3}\frac{a_2(p)}{p}\Big)^k + \Big(\sum_{p<X^3} a_2(p)\Big)^k\Big\}$$

$$\ll \frac{T}{(\log X)^{mk}}\,,$$

where we put

$$a_2(p) = \frac{\Lambda(p)-\Lambda_X(p)}{(\log p)^{m+1}}$$

and

$$b_2(n) = \sum_{p_1 p_2 \cdots p_k = n} a_2(p_1)a_2(p_2)\cdots a_2(p_k)$$
$$\ll \frac{1}{(\log X)^{mk}} .$$

The other two integrals can be treated in a similar manner and we get

$$U_{5,0} \ll \frac{T}{(\log X)^{mk}} .$$

For $\nu \neq 0$, we have

$$U_{5,\nu} \ll \left(\int_0^T \left(\sigma_{X,t} - \frac{1}{2}\right)^{4k\nu} dt\right)^{\frac{1}{2}} \left(\int_0^T \left| \sum_{p<X^3} \frac{\Lambda(p) - \Lambda_X(p)}{p^{\frac{1}{2}+it}(\log p)^{m+1-\nu}} \right|^{4k} dt\right)^{\frac{1}{2}}$$
$$+ \left(\int_0^T (\sigma_{X,t} - \frac{1}{2})^{4k\nu} dt\right)^{\frac{1}{2}} \left(\int_0^T \left| \sum_{p^2<X^3} \frac{\Lambda(p^2) - \Lambda_X(p^2)}{p^{1+i2t}(\log p)^{m+1-\nu}} \right|^{4k} dt\right)^{\frac{1}{2}}$$
$$+ \int_0^T \left(\sigma_{X,t} - \frac{1}{2}\right)^{2k\nu} \frac{1}{(\log X)^{2k(m-\nu)}} dt .$$

In the same manner as above, we get

$$\int_0^T \left| \sum_{p<X^3} \frac{\Lambda(p) - \Lambda_X(p)}{p^{\frac{1}{2}+it}(\log p)^{m+1-\nu}} \right|^{4k} dt = \int_0^T \left| \sum_{X^{2k}<n<X^{6k}} \frac{b_3(n)}{\sqrt{n} n^{it}} \right|^2 dt$$
$$\ll \sum_{X^{2k}<n<X^{6k}} (T + O(n)) \frac{|b_3(n)|^2}{n}$$
$$\ll \frac{1}{(\log X)^{2k(m-\nu)}} \left(T\left(\sum_{X<p<X^3} \frac{a_3(p)}{p}\right)^{2k} + \left(\sum_{X<p<X^3} a_3(p)\right)^{2k}\right)$$
$$\ll \frac{T}{(\log X)^{2k(m-\nu)}} ,$$

where we put

$$b_3(n) = \sum_{p_1 p_2 \cdots p_{2k} = n} a_3(p_1)a_3(p_2)\cdots a_3(p_{2k})$$
$$\ll \frac{1}{(\log X)^{2k(m-\nu)}}$$

with

$$a_3(p) = \frac{\Lambda(p) - \Lambda_X(p)}{(\log p)^{m+1-\nu}} .$$

Similarly, we get

$$\int_0^T \Bigg| \sum_{X<p^2<X^3} \frac{\Lambda(p^2)-\Lambda_X(p^2)}{p^{1+i2t}(\log p^2)^{m+1-\nu}} \Bigg|^{4k} dt \ll \frac{T}{(\log X)^{2k(m-\nu)}} .$$

Using Lemma 5, we get

$$\sum_{\nu=1}^{m} U_{5,\nu} \ll \frac{T}{(\log X)^{km}} .$$

Hence, we get

$$\sum_{\nu=0}^{m} U_{5,\nu} \ll \frac{T}{(\log X)^{km}} .$$

Since

$$\Bigg| \sum_{n<X^3} \frac{\Lambda_X(n)}{n^{it}(\log n)^{m+1-\nu}} \left(\frac{1}{n^{\sigma_{X,t}}}-\frac{1}{\sqrt{n}}\right) \Bigg| = \Bigg| \int_{\frac12}^{\sigma_{X,t}} \sum_{n<X^3} \frac{\Lambda_X(n)}{n^{\sigma+it}(\log n)^{m-\nu}} d\sigma \Bigg|$$

$$\leq \left(\sigma_{X,t}-\frac12\right) X^{\sigma_{X,t}-\frac12} \int_{\frac12}^{\infty} X^{\frac12-\sigma} \Bigg| \sum_{n<X^3} \frac{\Lambda_X(n)\log(Xn)}{n^{\sigma+it}(\log n)^{m-\nu}} \Bigg| d\sigma ,$$

we get

$$\begin{aligned} U_{6,\nu} &\ll \int_0^T \left(\sigma_{X,t}-\frac12\right)^{2k\nu+2k} X^{2k(\sigma_{X,t}-\frac12)} \\ &\quad \cdot \left(\int_{\frac12}^{\infty} X^{\frac12-\sigma} \Bigg| \sum_{n<X^3} \frac{\Lambda_X(n)\log(Xn)}{n^{\sigma+it}(\log n)^{m-\nu}} \Bigg| d\sigma \right)^{2k} dt \\ &\ll \left(\int_0^T \left(\sigma_{X,t}-\frac12\right)^{4k\nu+4k} X^{4k(\sigma_{X,t}-\frac12)} dt \right)^{\frac12} \\ &\quad \cdot \Bigg\{ \left(\int_0^T \left(\int_{\frac12}^{\infty} X^{\frac12-\sigma} \Bigg| \sum_{p<X^3} \frac{\Lambda_X(p)\log(Xp)}{p^{\sigma+it}(\log p)^{m-\nu}} \Bigg| d\sigma \right)^{4k} dt \right)^{\frac12} \\ &\quad + \left(\int_0^T \left(\int_{\frac12}^{\infty} X^{\frac12-\sigma} \Bigg| \sum_{p^2<X^3} \frac{\Lambda_X(p^2)\log(Xp^2)}{p^{2\sigma+i2t}(\log p)^{m-\nu}} \Bigg| d\sigma \right)^{4k} dt \right)^{\frac12} \\ &\quad + \left(\int_0^T \left(\int_{\frac12}^{\infty} X^{\frac12-\sigma} \log X d\sigma \right)^{4k} dt \right)^{\frac12} \Bigg\} . \end{aligned}$$

By Lemma 5 and Hölder inequality, we have

$$U_{6,\nu} \ll \frac{\sqrt{T}\sqrt{\log X}}{(\log X)^{2k\nu}}\left\{\left(\int_{\frac{1}{2}}^{\infty} X^{\frac{1}{2}-\sigma}\int_0^T \left|\sum_{p<X^3}\frac{\Lambda_X(p)\log(Xp)}{p^{\sigma+it}(\log p)^{m-\nu}\log^2 X}\right|^{4k} dt d\sigma\right)^{\frac{1}{2}}\right.$$

$$+\left(\int_{\frac{1}{2}}^{\infty} X^{\frac{1}{2}-\sigma}\int_0^T \left|\sum_{p^2<X^3}\frac{\Lambda_X(p^2)\log(Xp^2)}{p^{2\sigma+i2t}(\log p^2)^{m-\nu}\log^2 X}\right|^{4k} dt d\sigma\right)^{\frac{1}{2}}$$

$$\left.+\frac{\sqrt{T}}{\sqrt{\log X}(\log X)^{2k}}\right\}.$$

In the same manner as above, we get

$$\int_0^T \left|\sum_{p<X^3}\frac{\Lambda_X(p)\log(Xp)}{p^{\sigma+it}(\log p)^{m-\nu}\log^2 X}\right|^{4k} dt = \int_0^T \left|\sum_{n<X^{6k}}\frac{b_4(n)}{\sqrt{n}n^{it}}\right|^2 dt$$

$$\ll \sum_{n<X^{6k}}\frac{\mid b_4(n)\mid^2}{n}(T+O(n)),$$

where we put

$$b_4(n) = \sum_{p_1p_2\cdots p_{2k}=n} a_4(p_1)a_4(p_2)\cdots a_4(p_{2k})$$

with

$$a_4(p) = \frac{\Lambda_X(p)\log(Xp)}{p^{\sigma-\frac{1}{2}}(\log p)^{m-\nu}\log^2 X}.$$

If $m-\nu \geq 1$, then

$$a_4(p) \ll \frac{1}{(\log p)^{m-\nu-1}\log X}$$

and

$$b_4(n) \ll \frac{1}{\log^{2k} X}.$$

Thus in this case, we have

$$\sum_{n<X^{6k}}\frac{\mid b_4(n)\mid^2}{n} \ll \frac{1}{(\log X)^{4k}}\left(\sum_{p<X^3}\frac{1}{p(\log p)^{m+\nu-1}}\right)^{2k}$$

$$\ll \frac{1}{(\log X)^{4k}}\begin{cases}(\log\log X)^{2k} & \text{if } \nu = m-1\\ 1 & \text{if } \nu \leq m-2.\end{cases}$$

If $\nu = m$, then we have

$$\sum_{n<X^{6k}}\frac{|b_4(n)|^2}{n} \ll \frac{1}{(\log X)^{2k}}\left(\sum_{p<X^3}\frac{\log p}{p}\right)^{2k} \ll 1.$$

Consequently, we get

$$\int_0^T \left| \sum_{p<X^3} \frac{\Lambda_X(p)\log(Xp)}{p^{\sigma+it}(\log p)^{m-\nu}\log^2 X} \right|^{4k} dt \ll T \cdot \begin{cases} 1 & \text{if } \nu = m \\ \dfrac{(\log\log X)^{2k}}{(\log X)^{4k}} & \text{if } \nu = m-1 \\ \dfrac{1}{(\log X)^{4k}} & \text{if } \nu \le m-2\,. \end{cases}$$

Treating the other integral in a similar manner, we get

$$\sum_{\nu=0}^{m} U_{6,\nu} \ll \frac{T}{(\log X)^{2km}} + T\frac{(\log\log X)^k}{(\log X)^{2km}} + T\sum_{\nu=0}^{m-2} \frac{1}{(\log X)^{2k(\nu+1)}} \ll \frac{T}{(\log X)^{2k}}\,.$$

Similarly, we get

$$\begin{aligned} U_7 &= \int_0^T \left(\sigma_{X,t} - \frac{1}{2}\right)^{2k(m+1)} \left| \sum_{p<X^3} \frac{\Lambda_X(p)}{p^{\sigma_{X,t}+it}} \right|^{2k} dt \\ &\quad + \int_0^T \left(\sigma_{X,t} - \frac{1}{2}\right)^{2k(m+1)} \left| \sum_{p^2<X^3} \frac{\Lambda_X(p^2)}{p^{2\sigma_{X,t}+i2t}} \right|^{2k} dt \\ &\quad + \int_0^T \left(\sigma_{X,t} - \frac{1}{2}\right)^{2k(m+1)} dt \\ &\ll \int_0^T \left(\sigma_{X,t} - \frac{1}{2}\right)^{2k(m+1)} X^{2k\left(\sigma_{X,t}-\frac{1}{2}\right)} \\ &\quad \cdot \left(\int_{\frac{1}{2}}^{\infty} X^{\frac{1}{2}-\sigma} \left| \sum_{p<X^3} \frac{\Lambda_X(p)\log(Xp)}{p^{\sigma+it}} \right| d\sigma \right)^{2k} dt \\ &\quad + \int_0^T \left(\sigma_{X,t} - \frac{1}{2}\right)^{2k(m+1)} (\log X)^{2k} dt \\ &\ll \frac{T}{(\log X)^{2km}}\,. \end{aligned}$$

Combining all of the estimates, we get

$$\int_{T_1}^T \left(I_m(t) - \frac{1}{\pi}\Im\left\{ i^m \sum_{n<X^3} \frac{\Lambda(n)}{n^{\frac{1}{2}+it}(\log n)^{m+1}} \right\} \right)^{2k} dt \ll \frac{T}{(\log X)^{2k}}\,.$$

Finally, we have

$$\int_0^T I_m^{2k}(t)dt = \int_{T_1}^T I_m^{2k}(t)dt + O(\sqrt{X}(\log T)^{2k})\,,$$

where to treat the integral over the interval $0 \le t \le T_1$, we have used the estimate (cf. Theorem 1 on p. 4 of Fujii [5])

$$I_m(t) \ll \log t$$

for $t > t_o$ and any integer $m \ge 1$. The first integral can be treated as follows.

$$\begin{aligned}
\int_{T_1}^{T} I_m^{2k}(t)dt &= \int_{T_1}^{T} \left(\frac{1}{\pi}\Im\left\{i^m \sum_{n<X^3} \frac{\Lambda(n)}{n^{\frac{1}{2}+it}(\log n)^{m+1}}\right\}\right)^{2k} dt \\
&\quad + O\left(\int_{T_1}^{T} \left|\left(I_m(t) - \frac{1}{\pi}\Im\left\{i^m \sum_{n<X^3} \frac{\Lambda(n)}{n^{\frac{1}{2}+it}(\log n)^{m+1}}\right\}\right|^{2k} dt\right) \\
&\quad + O\left(\left(\int_{T_1}^{T} \left|\left(I_m(t) - \frac{1}{\pi}\Im\left\{i^m \sum_{n<X^3} \frac{\Lambda(n)}{n^{\frac{1}{2}+it}(\log n)^{m+1}}\right\}\right|^{2k} dt\right)^{\frac{1}{2k}}\right. \\
&\quad \left.\cdot \left(\int_{T_1}^{T} \left(\frac{1}{\pi}\Im\left\{i^m \sum_{n<X^3} \frac{\Lambda(n)}{n^{\frac{1}{2}+it}(\log n)^{m+1}}\right\}\right)^{2k} dt\right)^{1-\frac{1}{2k}}\right) \\
&= \int_{0}^{T} \left(\frac{1}{\pi}\Im\left\{i^m \sum_{n<X^3} \frac{\Lambda(n)}{n^{\frac{1}{2}+it}(\log n)^{m+1}}\right\}\right)^{2k} dt \\
&\quad + O\left(\int_{0}^{T_1} \left(\frac{1}{\pi}\Im\left\{i^m \sum_{n<X^3} \frac{\Lambda(n)}{n^{\frac{1}{2}+it}(\log n)^{m+1}}\right\}\right)^{2k} dt\right) + O\left(\frac{T}{(\log X)^{2k}}\right) \\
&\quad + O\left(\frac{T^{\frac{1}{2k}}}{(\log X)} \cdot \left(\int_{0}^{T} \left(\frac{1}{\pi}\Im\left\{i^m \sum_{n<X^3} \frac{\Lambda(n)}{n^{\frac{1}{2}+it}(\log n)^{m+1}}\right\}\right)^{2k} dt\right)^{1-\frac{1}{2k}}\right).
\end{aligned}$$

Now we have

$$\begin{aligned}
&\int_{0}^{T} \left(\frac{1}{\pi}\Im\left\{i^m \sum_{n<X^3} \frac{\Lambda(n)}{n^{\frac{1}{2}+it}(\log n)^{m+1}}\right\}\right)^{2k} dt \\
&\quad = \left(\frac{1}{2\pi i}\right)^{2k} \sum_{j=0}^{2k} \binom{2k}{j} \int_0^T \left(i^m \sum_{n<X^3} \frac{\Lambda(n)}{n^{\frac{1}{2}+it}(\log n)^{m+1}}\right)^{j} \\
&\quad \cdot \left(-(-i)^m \sum_{n<X^3} \frac{\Lambda(n)}{n^{\frac{1}{2}-it}(\log n)^{m+1}}\right)^{2k-j} dt \\
&\quad = \left(\frac{1}{2\pi i}\right)^{2k} \sum_{j=0}^{2k} \binom{2k}{j} i^{jm}(-1)^{2k-j}(-i)^{m(2k-j)} \\
&\quad \cdot \sum_{n_1,\cdots,n_j<X^3} \frac{\Lambda(n_1)\cdots\Lambda(n_j)}{(n_1\cdots n_j)^{\frac{1}{2}}(\log n_1\cdots\log n_j)^{m+1}}
\end{aligned}$$

$$\cdot \sum_{n_{j+1},\cdots,n_{2k}<X^3} \frac{\Lambda(n_{j+1})\cdots\Lambda(n_{2k})}{(n_{j+1}\cdots n_{2k})^{\frac{1}{2}}(\log n_{j+1}\cdots\log n_{2k})^{m+1}}$$

$$\cdot \int_0^T \left(\frac{n_{j+1}\cdots n_{2k}}{n_1\cdots n_j}\right)^{it} dt$$

$$= \frac{T}{(2\pi)^{2k}} \sum_{j=0}^{2k} \binom{2k}{j} (-1)^{(j+k)(m+1)}$$

$$\cdot \sum_{\substack{n_1,\cdots,n_{2k}<X^3 \\ n_1\cdots n_j = n_{j+1}\cdots n_{2k}}} \frac{\Lambda(n_1)\cdots\Lambda(n_{2k})}{n_1\cdots n_j(\log n_1\cdots\log n_{2k})^{m+1}}$$

$$+ O\Biggl(\sum_{j=0}^{2k} \sum_{\substack{n_1,\cdots,n_{2k}<X^3 \\ n_1\cdots n_j \neq n_{j+1}\cdots n_{2k}}} \frac{\Lambda(n_1)\cdots\Lambda(n_{2k})}{\sqrt{n_1\cdots n_{2k}}(\log n_1\cdots\log n_{2k})^{m+1}\left|\log\frac{n_1\cdots n_j}{n_{j+1}\cdots n_{2k}}\right|}\Biggr)$$

$$= \frac{T}{(2\pi)^{2k}} \sum_{j=0}^{2k} \binom{2k}{j} (-1)^{(j+k)(m+1)}$$

$$\cdot \sum_{\substack{n_l=2 \\ n_1\cdots n_j = n_{j+1}\cdots n_{2k}}}^{\infty} \frac{\Lambda(n_1)\cdots\Lambda(n_{2k})}{n_1\cdots n_j(\log n_1\cdots\log n_{2k})^{m+1}} + O(T^{1-\varepsilon}).$$

Consequently, we get

$$\int_0^T I_m^{2k}(t)dt = \frac{T}{(2\pi)^{2k}} \sum_{j=0}^{2k} \binom{2k}{j} (-1)^{(j+k)(m+1)}$$

$$\cdot \sum_{\substack{n_l=2 \\ n_1\cdots n_j = n_{j+1}\cdots n_{2k}}}^{\infty} \frac{\Lambda(n_1)\cdots\Lambda(n_{2k})}{n_1\cdots n_j(\log n_1\cdots\log n_{2k})^{m+1}}$$

$$+ O\left(\frac{T}{\log T}\right).$$

This proves Theorem 2.

Combining this result for $k = 1$ with the previous evaluations of U_2 and U_3 in the section 2, we get

$$\int_0^T \tilde{S}_m^2(t)dt = \frac{T}{2\pi^2} \sum_{n=2}^{\infty} \frac{\Lambda^2(n)}{n(\log n)^{2(m+1)}}$$

$$+ 4 \sum_{\substack{h+2r=m \\ r\geq 1, h\geq 0}} (-1)^{r-1} \frac{1}{(2r)!\cdot h!} \sideset{}{'}\sum_{\substack{\beta+i\gamma \\ \beta>\frac{1}{2}, 0<\gamma<T}} \left(\beta - \frac{1}{2}\right)^{2r}$$

$$
\begin{aligned}
&\cdot \left\{ \sum_{j=0}^{h-1} (-1)^j \frac{h!}{(h-j)!} I_{m+1+j}(T)(T-\gamma)^{h-j} \right. \\
&+ (-1)^h h! \cdot (I_{m+1+h}(T) - I_{m+1+h}(\gamma)) \\
&\left. + \sum_{j=0}^{h} (-1)^j \frac{h!}{(h-j)!} \cdot 2 \cdot \chi_0(m+j) \cdot (-1)^{\frac{m+j-1}{2}} H(m+1+j, h-j; T, \gamma) \right\} \\
&+ 4 \sum_{\substack{h_1+2r_1=m \\ r_1 \geq 1, h_1 \geq 0}} \sum_{\substack{h_2+2r_2=m \\ r_2 \geq 1, h_2 \geq 0}} (-1)^{r_1+r_2} \frac{1}{(2r_1)! \cdot h_1!} \frac{1}{(2r_2)! \cdot h_2!} \\
&\cdot \sideset{}{'}\sum_{\substack{\beta_1+i\gamma_1 \\ \beta_1 > \frac{1}{2}, 0<\gamma_1<T}} \sideset{}{'}\sum_{\substack{\beta_2+i\gamma_2 \\ \beta_2 > \frac{1}{2}, 0<\gamma_2<T}} \left(\beta_1 - \frac{1}{2}\right)^{2r_1} \left(\beta_2 - \frac{1}{2}\right)^{2r_2} \\
&\cdot \sum_{l=0}^{h_1+h_2} (-1)^l \frac{T^{l+1} - (\max(\gamma_1, \gamma_2))^{l+1}}{l+1} \sum_{\substack{0 \leq j \leq h_1, 0 \leq k \leq h_2 \\ j+k=h_1+h_2-l}} \binom{h_1}{j} \binom{h_2}{k} \gamma_1^j \gamma_2^k \\
&+ O\left(\frac{T}{\log T}\right).
\end{aligned}
$$

This proves Theorem 1.

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